

MINIMUM UNCERTAINTY AND SQUEEZING IN DIFFUSION PROCESSES AND STOCHASTIC QUANTIZATION

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Abstract

We show that uncertainty relations, as well as minimum uncertainty coherent and squeezed states, are structural properties for diffusion processes. Through Nelson stochastic quantization we derive the stochastic image of the quantum mechanical coherent and squeezed states.

1 Introduction

It is well known that the theory of stochastic processes is a powerful tool in the study of the interplay between probabilistic and deterministic evolution [1]. In quantum mechanics, and in particular in quantum optics, such interplay is expressed by the states of minimum uncertainty, the coherent [2] and squeezed states [3], which are viewed as the "most classical" states.

In this paper we report on a recent derivation [4] of uncertainty relations for classical stochastic processes of the diffusion type, and we determine the diffusion processes of minimum uncertainty (MUDPs). We find that a special class among them is associated to Gaussian probability distributions with time-conserved covariance and mean value with classical time evolution: we refer to them as strictly coherent MUDPs. We will also identify Gaussian MUDPs with time-dependent covariance and conserved expectation value: we refer to them as broadly coherent MUDPs. By exploiting Nelson's stochastic quantization scheme [5], we will show that the strictly coherent MUDPs provide the stochastic image of the standard quantum mechanical coherent and squeezed coherent states, while the broadly coherent MUDPs are associated with the phenomenon of time-dependent squeezing.

Our study is motivated by the possibility that the formalism of stochastic processes offers to treat on the same footing, in a unified mathematical language, the interplay between fluctuations of different nature, for instance quantum and thermal [6].

Beyond the case of diffusion processes, it is interesting to note that coherence and squeezing have recently emerged in other contexts wider than quantum mechanics ([7], [8]).

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2 Uncertainty and Coherence in Diffusion Processes

In what follows, without lack of generality, we will consider a one-dimensional random variable q . The associated diffusion process $q(t)$ obeys

$$dq(t) = v_{(+)}(q(t), t)dt + \nu^{1/2}(q(t), t)dw(t), \quad dt > 0, \quad (1)$$

where $v_{(+)}(q(t), t)$, is the forward drift, $\nu(q(t), t)$ is the diffusion coefficient, and $dw(t)$ is a Gaussian white noise, superimposed on the otherwise deterministic evolution, with expectation $E(dw(t)) = 0$ and covariance $E(dw^2(t)) = 2dt$. The forward and the backward drifts $v_{(+)}(x, t)$ and $v_{(-)}(x, t)$ are defined as

$$\begin{aligned} v_{(+)}(x, t) &= \lim_{\Delta t \rightarrow 0^+} E \left(\frac{q(t + \Delta t) - q(t)}{\Delta t} \mid q(t) = x \right), \\ v_{(-)}(x, t) &= \lim_{\Delta t \rightarrow 0^+} E \left(\frac{q(t) - q(t - \Delta t)}{\Delta t} \mid q(t) = x \right). \end{aligned} \quad (2)$$

The definitions of $v_{(+)}$ and $v_{(-)}$ are not independent, but related by [5]

$$v_{(-)}(x, t) = v_{(+)}(x, t) - \frac{2\partial_x(\nu(x, t)\rho(x, t))}{\rho(x, t)}. \quad (3)$$

It is now convenient to define the osmotic velocity $u(x, t)$ and the current velocity $v(x, t)$

$$\begin{aligned} u(x, t) &= \frac{v_{(+)}(x, t) - v_{(-)}(x, t)}{2} = \frac{\partial_x(\nu(x, t)\rho(x, t))}{\rho(x, t)}, \\ v(x, t) &= \frac{v_{(+)}(x, t) + v_{(-)}(x, t)}{2}. \end{aligned} \quad (4)$$

From the former definitions it is clear that $u(x, t)$ “measures” the non-differentiability of the random trajectories, controlling the degree of stochasticity. In the deterministic limit u vanishes, and $v(x, t)$ goes to the classical velocity $v(t)$.

Finally, we have the continuity equation

$$\partial_t \rho(x, t) = -\partial_x(\rho(x, t)v(x, t)). \quad (5)$$

It is straightforward to check that $E(v_{(+)}) = E(v_{(-)}) = E(v)$, and $E(u) = 0$. Further,

$$E(v) = \frac{d}{dt}E(q) \quad \forall t. \quad (6)$$

For the product qu , we have $|E(qu(q, t))| = E(\nu(q, t))$. By Schwartz’s inequality, the r.m.s. deviations Δq and Δu satisfy

$$\Delta q \Delta u \geq E(\nu(q, t)). \quad (7)$$

Inequality (7) is the uncertainty relation for any diffusion process. Equality in (7) defines the MUDPs. Saturation of Schwartz's inequality yields $u(x, t) = C(t)(x - E(q))$, where $C(t)$ is an arbitrary function of time. Considering constant ν and time-dependent ν , in both cases we obtain a Gaussian minimum uncertainty density:

$$\rho(\hat{x}, t) = \frac{1}{\sqrt{2\pi(\Delta q)^2}} \exp \left[-\frac{(x - E(q))^2}{2(\Delta q)^2} \right], \quad (8)$$

where $2(\Delta q)^2 = -\nu(t)/C(t)$.

From eq. (5) we can determine the current velocity:

$$v(x, t) = \frac{d}{dt} E(q) + \frac{1}{\Delta q} \left(\frac{d}{dt} \Delta q \right) F(x, t), \quad (9)$$

where

$$F(x, t) = x - E(q) + E(q) \exp \left[\frac{x^2 - 2xE(q)}{2(\Delta q)^2} \right]. \quad (10)$$

Eqs. (8)-(10) lead to the stochastic differential equation obeyed by any MUDP:

$$dq(t) = [A(t) + B(t)q(t)]dt + \nu^{1/2}(t)dw(t). \quad (11)$$

It is interesting to observe that (11) defines the so-called *linear processes in narrow sense*. When $A(t) = 0$ they are the time-dependent Ornstein-Uhlenbeck processes. These last ones play a natural role in the theory of low noise systems [1], which are thus found to be related with MUDPs.

The possible choices of $E(q)$ and Δq in (8) are not independent: taking the expectation value of in (9)-(10), and reminding (6) one has that either

$$\begin{cases} \Delta q &= \text{const.} \quad \forall t, \\ E(q) &= j(t), \end{cases} \quad (12)$$

or

$$\begin{cases} \Delta q &= k(t), \\ E(q) &= 0 \quad \forall t, \end{cases} \quad (13)$$

where $j(t)$ and $k(t)$ are arbitrary functions of time, and we have chosen for simplicity $q(t = 0) = 0$. Consider first case (12): Δq does not spread; also, it is immediate to verify that the expectation value of the process $E(q)$ follows a classical trajectory:

$$v(x, t) = \frac{d}{dt} E(q) = v(t), \quad (14)$$

As a consequence, MUDPs of the form (8) obeying (12) and (14) are coherent in a sense precisely analogous to that of quantum mechanical coherent states: we will refer to them as strictly coherent MUDPs and to processes (8) obeying (13) as broadly coherent MUDPs.

It is possible to discriminate on physical grounds the strictly coherent MUDPs from the broadly coherent ones by observing that (12) and (14) come as immediate consequence on imposing the Ehrenfest condition

$$v(E(q), t) = \frac{d}{dt} E(q), \quad (15)$$

so that the strictly coherent MUDPs can be viewed as the most deterministic semi-classical processes.

Consider the scale transformation $x \rightarrow e^{-s}x$ which automatically implies $u \rightarrow e^s u$, where s is the scale parameter. The Gaussian distribution (8) is form-invariant under this transformation, while the uncertainty product (7) is strictly invariant with $\Delta q \rightarrow e^{-s}\Delta q$ and $\Delta u \rightarrow e^s\Delta u$. We will show next that in the framework of Nelson stochastic quantization this transformation is just the squeezing transformation of quantum mechanics. In this context, broadly coherent MUDPs are of special interest when considering time-dependent squeezing.

3 Nelson Diffusions

A very important class of diffusion processes (Nelson diffusions) in physics has been introduced by Nelson in his stochastic formulation of quantum mechanics [5].

To each single-particle quantum state $\Psi = \exp \left[R + \frac{i}{\hbar} S \right]$, Nelson stochastic quantization associates the diffusion process $q(t)$ with

$$\nu = \frac{\hbar}{2m}, \quad \rho(x, t) = |\Psi(x, t)|^2, \quad v(x, t) = \frac{1}{m} \frac{\partial S(x, t)}{\partial x}, \quad (16)$$

where m is the mass of the particle. At the dynamical level, the Schroedinger equation with potential $V(x, t)$ is equivalent to the Hamilton-Jacobi-Madelung equation

$$\partial_t S(x, t) + \frac{(\partial_x S(x, t))^2}{2m} - \frac{\hbar^2}{2m} \frac{\partial_x^2 \rho^{1/2}(x, t)}{\rho^{1/2}(x, t)} = -V(x, t). \quad (17)$$

It is well known [9] that for Nelson diffusions the uncertainties Δq and Δu are related to the quantum mechanical uncertainties $\Delta \hat{q}$ and $\Delta \hat{p}$ of the position and momentum operators \hat{q} and \hat{p} by

$$\Delta \hat{q} = \Delta q, \quad (\Delta \hat{p})^2 = m[(\Delta u)^2 + (\Delta v)^2], \quad (18)$$

$$(\Delta \hat{q})^2 (\Delta \hat{p})^2 \geq (\Delta q)^2 (\Delta m u)^2 \geq \frac{\hbar^2}{4}.$$

Minimum uncertainty Nelson diffusions (MUNDs) are MUDPs. Correspondingly, we will speak of strictly and broadly coherent MUNDs. By solving (17) for MUNDs we obtain $V(x, t)$ and the classical equations of motion for $E(q)$. For strictly coherent MUNDs (12) we have

$$V(x, t) = \frac{m}{2} \omega^2 x^2 + f(t)x + V_0(t), \quad \omega^2 = \frac{\hbar^2}{4m^2(\Delta q)^4}, \quad (19)$$

$$\frac{d^2}{dt^2} E(q) + \omega^2 E^2(q) = f(t).$$

When the arbitrary constants $f(t)$ and $V_0(t)$ vanish, eqs. (19) are those of the classical harmonic oscillator and the associated quantum states are the standard Glauber coherent states; when $f(t) = \text{const.}$ we have the Klauder-Sudarshan displaced oscillator coherent states; finally, when $f(t)$ is truly time-dependent, we obtain the Klauder-Sudarshan driven oscillator coherent states [10].

For broadly coherent MUNDs we have instead

$$V(x, t) = \frac{1}{2} m \omega^2(t) x^2 + V_0(t), \quad \omega^2(t) = \dot{g}(t) + 2g^2(t) - \frac{\hbar^2}{8m^2(\Delta q)^4}, \quad (20)$$

where $g(t) = (\Delta q)^{-1} d\Delta q/dt$ must be such that $\omega^2(t)$ is positive. Eq. (20) describes the parametric oscillator potential, associated to the feature of time-dependent squeezing.

Furthermore, we can identify among MUNDs those corresponding either to Heisenberg or to Schroedinger minimum uncertainty. The key relation, easy to prove, is

$$E(vq) - E(v)E(q) = \frac{\langle \{\hat{Q}, \hat{P}\} \rangle_\psi}{2}; \quad \hat{Q} = \hat{q} - \langle \hat{q} \rangle_\psi, \quad \hat{P} = \frac{\hat{p} - \langle \hat{p} \rangle_\psi}{m}, \quad (21)$$

where $\langle \{\cdot, \cdot\} \rangle_\psi$ denotes the expectation of the anti-commutator in the state Ψ , i.e. the Schroedinger part of the quantum mechanical uncertainty.

Eqs. (18) and (21) show that the strictly coherent MUNDs (19) exhaust the Heisenberg minimum uncertainty states, while the broadly coherent MUNDs (20) form a subset of the Schroedinger minimum uncertainty states.

Finally, we investigate the possibility of letting ν be time-dependent in the context of quantum mechanics. From the first of equations (16) this means letting either m or \hbar be functions of time.

This latter case seems a bit speculative at this stage. We thus fix our attention on the case of time-dependent mass $m(t)$ and constant \hbar .

For such systems it can be immediately verified that the Nelson scheme (16)-(17) still holds with $m(t)$ replacing m . Considering the most interesting case of strictly coherent MUNDs, which means choosing $C(t) \propto \nu(t)$, and solving (17) we obtain

$$V(x, t) = \frac{1}{2} m(t) \omega^2(t) x^2 + f(t)x + V_0(t), \quad \omega^2(t) = \frac{\hbar^2}{4m^2(t)(\Delta x)^4}, \quad (22)$$

$$\frac{d^2}{dt^2} E(q) + \frac{\dot{m}(t)}{m(t)} \frac{d}{dt} E(q) + \omega^2(t) E(q) = \frac{f(t)}{m(t)},$$

where $f(t)$, $V_0(t)$ are arbitrary functions of time. Eqs. (22) supplemented with $m(t) = m_0 e^{\Gamma(t)}$ define the dynamics of the damped parametric oscillator. The stochastic approach thus sheds new light in a unified treatment on the study of quantum dissipative oscillators, for it allows to derive for the expectation value the dynamical equation (22) that was so far unknown.

In conclusion, we have shown that the quantum mechanical concepts of uncertainty, coherence, and squeezing can be imported in the probabilistic arena of diffusion processes. This appears to be possible because of a subtle interplay between fluctuations, control, and optimization. Conversely, we may also say that these features of quantum mechanics can be traced back and related to general properties of diffusion processes.

Work on this subject is in progress, and includes application of our scheme to polymer dynamics and chemical reactions, uncertainty relations in field theory and dynamical systems on lattices and manifolds.

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SECTION 6

FIELD THEORY AND GENERAL INTEREST

